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Stochastic Representation of Nearly-Gaussian, Nonlinear Processes

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The use of polynomial functionals of the white noise process is discussed for the treatment of nonlinear random processes. It is noted that such treatments are useful for nearly-Gaussian processes. Applications of such representations to nonlinear systems and to nonlinear fluid mechanics problems (turbulence) are reviewed.

KEY WORDS: Nonlinear Processes; Stochastic; Turbulence; Wiener.

1. INTRODUCTION

The representation to be discussed in this paper is based upon the white noise process. This process was used in early work in connection with discussions of Brownian motion. (For a review of this work one can refer to Wiener.[43] As will be set forth below, the white noise process is used in polynomial functionals to represent Gaussian and non-Gaussian processes. Perhaps the earliest work on these representations is that of Cameron and Martin.[7] Heuristic extensions of these representations to deal with vector processes have been given by Imamura, Meecham and Siegel,[19] and more recently, mathematically rigorous extensions have been made by Bergman[4] in conjunction with Professor J. Bass.

Much work has been done on the application of these stochastic representations to filter theory and circuit theory (see, e.g., Van Trees,[40] Lee and Schetzen,[26] and Schetzen[36]). Similarly there has been a large body of work devoted to the application of these representations to fluid flow problems including magnetohydrodynamic turbulence problems (Meecham and Siegel,[27] Imamura, Meecham, and Siegel,[19] Nihoul,[31] Meecham and Jeng,[28] Saffman[35]). There has also been much effort directed toward the use of these representations in other nonlinear physical processes (Barrett,[1] Gyftopoulos and Hooper,[14,15] Harris and Lapidus,[16]

[1] University of California, Los Angeles, and RAND Corporation, Santa Monica, California.

Vassilopoulos,[41] Kyong and Gyftopoulos[25]). The various results of these treatments and some of the difficulties which have arisen are discussed more fully in succeeding sections.

Parallel to this work dealing with applications of the representation to nonlinear physical problems has been mathematical work devoted to developing the theory on a rigourous basis. In addition to the papers already cited in this connection, one could mention, e.g., the work of Ito and Nisio.[20] Cameron has continued to develop the mathematical aspects of the theory (see, e.g., Reference 8) in conjunction with associates of his. Our emphasis here will be on the use of the stochastic representation in dealing with physical problems. The reader is referred to the references given for the mathematical work which is presently available.

It is the purpose of this paper to show the use of the stochastic representation in statistical physics problems or at least to show those applications with which the author is familiar. It is recognized that these methods have been developed fairly recently and consequently a review will be helpful. To start our discussion, suppose we consider a single random variable a which is Gaussian with unit variance. For a physical application consider the displacement of a mass in a one-dimensional mass–spring system when the initial displacement is zero, and when the initial velocity of the mass is *Gaussianly* distributed with a given variance. The equation of motion is

$$\ddot{X} + \omega^2 X = 0 \tag{1}$$

In terms of our random variables this has a solution at later time

$$X(t) = (V_0'/\omega)a \sin \omega t \tag{2}$$

where $(V_0')^2$ is the variance of $\dot{X}(0)$, of the initial velocity. The single random element a suffices for this simple problem. Suppose now that the initial velocity has non-Gaussian components. We can represent this initial velocity in terms of a polynomial series involving the random element a. It is convenient to use Hermite polynomials, and we have

$$H^{(0)} \equiv 1, \qquad H^{(1)} \equiv a, \qquad H^{(2)} \equiv a^2 - 1, \qquad H^{(3)} = a^{(3)} - 3^a, \ldots \tag{3}$$

Note that these separate terms may be said to be statistically orthogonal. That is, using the properties of the polynomials we have

$$\langle H^{(i)} H^{(j)} \rangle = i!\, \delta_{ij}$$

The initial random velocity of the particle may be written in terms of functions of a with coefficients as shown

$$V_0 = \langle V_0 H^{(1)} \rangle H^{(1)} + \langle V_0 H^{(2)} \rangle \frac{1}{2!} H^{(2)} + \langle V_0 H^{(3)} \rangle \frac{1}{3!} H^{(3)} + \cdots \tag{4}$$

The brackets $\langle\ \rangle$ indicate ensemble averages throughout this work. The zero-order term would appear in this expansion if the average of the initial velocity were not zero. The solution of this problem is of course

$$X(t) = (V_0/\omega) \sin \omega t \tag{5}$$

where the random initial velocity V_0 is expressed in terms of the random process expansion given in (4). To find the basic, Gaussian random process $H^{(1)}$ (or a) we should have to invert Eq. (4). This can be done by successive approximation; that is, begin with the approximation

$$H^{(1)} \approx V_0/V_0' \tag{6}$$

and successively substitute for terms of progressively higher order in (4). Clearly, the problem discussed here is trivial and requires no advance formalism for its treatment. It serves merely to define quantities analogous to those to be used below. It will be seen that these concepts, after appropriate generalization, are indeed useful in the treatment of nonlinear random processes.

Throughout this paper we use distributions where necessary and assume where needed that all functions are sufficiently well-behaved so that we may interchange limits. It sould be said that the notation and details of treatment have not yet become uniform in the literature. The author adopts here a notation used by many but not all workers.

2. REVIEW OF THE NONLINEAR RANDOM REPRESENTATION

In this section we review the various generalizations of representations of the type given in (3) above. In later sections we treat the use of these representations in the examination of physical processes. First of all, consider those discrete processes involving more than one random variable. Use for an example the random-initial-value three-mode problem. This problem is defined as follows (for a discussion see Kraichnan,[24] Orszag and Bissonnette[33]):

$$\dot{X}_i = A_i X_j X_k \tag{7}$$

with

$$A_1 + A_2 + A_3 = 0$$

where A_i are constant (the condition guarantees that the sum of the variances is constant). Here i, j, k are any permutation of 1, 2, 3. For the treatment of such problems as this we generalize the representation (3) to include three independent Gaussian random processes,

$$\begin{aligned} &H_i^{(1)} \quad \text{with} \quad \langle H_i^{(1)} H_j^{(1)} \rangle = \delta_{ij} \\ &H_{ij}^{(2)} = H_i^{(1)} H_j^{(1)} - \delta_{ij} \\ &H_{ijk}^{(3)} = H_i^{(1)} H_j^{(1)} H_k^{(1)} - (H_i^{(1)} \delta_{jk} + H_j^{(1)} \delta_{ki} + H_k^{(1)} \delta_{ij}) \end{aligned} \tag{8}$$

Note that these quantities are also statistically orthogonal,

$$\langle H^{(i)} H^{(j)} \rangle = 0, \qquad i \neq j$$

The processes $X_i(t)$ may be represented at a given time by the expressions

$$X_i(t) = P_{i\alpha}(t)\, H_\alpha^{(1)} + Q_{i\alpha\beta}(t)\, H_{\alpha\beta}^{(2)} + \cdots \tag{9}$$

The coefficients P, Q,... are nonrandom functions of the time. In representations of this type the only random elements are the white noise processes. We suppose summation from 1 to 3 on repeated indices. In (9) the coefficients P, Q,... become the unknowns for an initial-value problem. The representation (9) may be substituted into (7) in order to determine these unknown functions. It is known that (7) has an equilibrium solution when

$$A_i\langle X_j^2\rangle\langle X_k^2\rangle + A_j\langle X_k^2\rangle\langle X_i^2\rangle + A_k\langle X_i^2\rangle\langle X_j^2\rangle = 0 \tag{10}$$

It may be verified that the first, Gaussian, term of (9) is sufficient to give statistical equilibrium if the variances satisfy (10).

This is a good place to emphasize the fact that representations of the type (9) are not unique though they are complete (for very general classes of processes). Wiener[43] emphasized the possibility of "measure-preserving transformations." To see this possibility, consider a single random process of the type (3) undergoing a discontinuous transformation as follows:

$$\begin{aligned} H^{(1)\prime} &= -H^{(1)}, \qquad |H^{(1)}| < 1 \\ &= H^{(1)}, \qquad |H^{(1)}| > 1 \end{aligned} \tag{11}$$

We could represent the random process V_0 in terms of $H^{(1)\prime}$ with a series similar to that given in (4) but with different coefficients, which may be calculated from (11). This lack of uniqueness in the representation means that the convergence of a series of the type (9) may be improved at later times by permitting both the coefficients (P, Q,...) and the white noise processes themselves to be functions of time. To guarantee that the properties of the representation do not change, it is sufficient to require of the transformation

$$\langle H^{(i)\prime}_{\alpha_1\alpha_2\cdots} H^{(j)\prime}_{\beta_1\beta_2\cdots}\rangle = \langle H^{(i)}_{\alpha_1\alpha_2\cdots} H^{(j)}_{\beta_1\beta_2\cdots}\rangle \tag{12}$$

for all functionals. Here $H^{(i)\prime}$ are the transformed functionals. Doi and Imamura[13] have discussed such transformations which are dependent upon a parameter (time).

There have been a number of recent attempts to improve the convergence characteristics of representations of the general type (9) by permitting the white noise process to be a function of time. The transformed functionals obey a restriction like (12) (see Bodner,[5] Canavan and Leith,[9] Clever[11]). In some problems it appears that transformations of the representation of this general type will prove most useful. It is possible in some cases to avoid the question of the explicit use of such transformations by formulating the problems in terms of moments (taken to be functions of time). The moments will necessarily be invariant under transformations of the representation, as can be seen from the basic condition (12). Such an approach is adequate for simultaneous correlations (moments), for example. However, to obtain

information about time correlations, we are forced to use the representation at different times and may be forced to employ the (time) transformed representations as well in order to obtain adequate convergence.

Consider the representation of a random time series using continuous functionals patterned after (3). We suppose that the basic white noise process is $H^{(1)}(t)$ with the covariance,

$$\langle H^{(1)}(t)\, H^{(1)}(t')\rangle = \delta(t - t') \tag{13}$$

with δ the Dirac delta function and with the higher-order moments those of a Gaussian process. This process can be thought of as the limit, in the sense of distributions, of a process defined in the following way. Suppose the t axis is broken up into discrete cells of width Δ. Imagine a set of statistically independent Gaussian processes, one in each cell Δ. Further suppose that the variance of each process is Δ^{-1}. Then, if we permit the width of the cell to approach zero, we approach the process described in (13). To represent non-Gaussian processes, we need functionals of the white noise process $H^{(1)}(t)$ as shown

$$\begin{aligned} &H^{(1)}(t) \\ &H^{(2)}(t_1, t_2) = H^{(1)}(t_1)\, H^{(1)}(t_2) - \delta(t_1 - t_2) \\ &H^{(3)}(t_1, t_2, t_3) = H^{(1)}(t_1)\, H^{(1)}(t_2)\, H^{(1)}(t_3) \\ &\qquad - [H^{(1)}(t_1)\, \delta(t_2 - t_3) + H^{(1)}(t_2)\, \delta(t_3 - t_1) + H^{(1)}(t_3)\, \delta(t_1 - t_2)] \end{aligned} \tag{14}$$

As above, these quantities are constructed so that they are statistically orthogonal:

$$\langle H^{(i)} H^{(j)}\rangle = 0, \qquad i \neq j \tag{15}$$

A Gaussian or non-Gaussian time series $y(t)$ may be represented in a way analogous to (9):

$$y(t) = \int P(t; t_1)\, H^{(1)}(t_1)\, dt_1 + \iint Q(t; t_1, t_2)\, H^{(2)}(t_1, t_2)\, dt_1\, dt_2 + \cdots \tag{16}$$

The functions P, Q,... are nonrandom quantities (usually unknowns to be determined). The only random element (as above) in the representation is $H^{(1)}(t)$. A quantity represented by (16) may in general be nonstationary. A statistically stationary process may be represented by using difference arguments as follows:

$$y(t) = \int P(t - t_1)\, H^{(1)}(t_1)\, dt_1 + \iint Q(t - t_1, t - t_2)\, H^{(2)}(t_1, t_2)\, dt_1\, dt_2 + \cdots \tag{17}$$

Such a representation would be useful, for example, for the representation of a process which is nearly Gaussian and is driven by a statistically stationary, Gausssian forcing term, the process having run long enough so that it has become stationary. Alternatively, upon replacing the arguments t by the independent space variable x, it will be seen that a representation of the type (16) or (17) is useful in the treatment of random-initial-value problems.

These representations can be generalized for the treatment of vector processes which are functions of vector arguments (see Reference 19). Define the white noise process $H_i^{(1)}(\mathbf{r})$. It has covariance

$$\langle H_i^{(1)}(\mathbf{r})\, H_j^{(1)}(\mathbf{r}')\rangle = \delta_{ij}\, \delta(\mathbf{r} - \mathbf{r}') \tag{18}$$

patterned after (13). The higher-order moments are those of Gaussian processes. This process is made up of a set of white noise processes, a different one for each Cartesian direction and each space point, all of which are statistically independent of one another. The variances of these independent Gaussian processes are given by (18). Following (14) above, we define the higher-order functionals as follows:

$$\begin{aligned}
&H_i^{(1)}(\mathbf{r}) \\
&H_{ij}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = H_i^{(1)}(\mathbf{r}_1)\, H_j^{(1)}(\mathbf{r}_2) - \delta_{ij}\, \delta(\mathbf{r}_1 - \mathbf{r}_2) \\
&H_{ijk}^{(3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&\quad = H_i^{(1)}(\mathbf{r}_1)\, H_j^{(1)}(\mathbf{r}_2)\, H_k^{(1)}(\mathbf{r}_3) \\
&\qquad - [H_i^{(1)}(\mathbf{r}_1)\, \delta_{jk}\, \delta(\mathbf{r}_2 - \mathbf{r}_3) + H_j^{(1)}(\mathbf{r}_2)\, \delta_{ki}\, \delta(\mathbf{r}_3 - \mathbf{r}_1) + H_k^{(1)}(\mathbf{r}_3)\, \delta_{ij}\, \delta(\mathbf{r}_1 - \mathbf{r}_2)]
\end{aligned} \tag{19}$$

A general random vector function is represented by integrals analogous to (16). Furthermore, if we know that the process is statistically homogeneous, we can represent it [see (17)] as follows:

$$\begin{aligned}
U_i(\mathbf{r}) = &\int U_{i\alpha}^{(1)}(\mathbf{r} - \mathbf{r}_1)\, H_\alpha^{(1)}(\mathbf{r}_1)\, d\mathbf{r}_1 \\
&+ \iint U_{i\alpha\beta}^{(2)}(\mathbf{r} - \mathbf{r}_1, \mathbf{r} - \mathbf{r}_2)\, H_{\alpha\beta}^{(2)}(\mathbf{r}_1, \mathbf{r}_2)\, d\mathbf{r}_1\, d\mathbf{r}_2 + \cdots
\end{aligned} \tag{20}$$

From the definitions (19) we see that we may require that the nonrandom coefficient functions U be symmetric

$$U_{i\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = U_{i\beta\alpha}(\mathbf{r}_2, \mathbf{r}_1), \text{ etc.} \tag{21}$$

It is possible to guarantee properties for the ensemble of functions $\mathbf{u}$ by building the corresponding properties into the coefficients $U^{(i)}$. For example, suppose that we require $\nabla \cdot \mathbf{u} = 0$. It can be guaranteed that every member of the ensemble given by (20) will have this property if we require

$$\frac{\partial}{\partial r_i} U_{i\alpha}(\mathbf{r}) = 0 \tag{22}$$

Further, consider processes whose moments depend only on the vector differences of their arguments for their tensor characteristics. (Such processes are ordinarily called statistically isotropic.) Statistical properties such as this can be built into the ensemble of $\mathbf{u}$ by requiring corresponding properties for the nonrandom kernel functions $U^{(i)}$.

Thus, for example, we can guarantee that the ensemble of functions $\mathbf{u}$ will be statistically isotropic if

$$\begin{aligned} U_{ij}^{(1)}(\mathbf{r}) &= F_1(r)\, r_i r_j + F_2(r)\, \delta_{ij} \\ U_{ijk}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) &= \text{sum of all combinations of } r_{1i},\ r_{2i}, \text{ and } \delta_{ij} \end{aligned} \tag{23}$$

Kahng and Siegel[23] habe extended the representation to include pseudo-tensor forms. Referring to (20), we see that if $\mathbf{u}$ is a vector, then we could ask that $U^{(1)}$ be a pseudo-tensor if at the same time the white noise process $H_\gamma^{(1)}(\mathbf{r}_1)$ is a pseudo-vector. Similarly, $U^{(2)}$ would be a tensor, as we see from (19) noting that $H_{\alpha\beta}^{(2)}$ is a true tensor. Higher-order terms have an analogous structure. Furthermore, if $\mathbf{u}$ were a psuedo-vector, we could represent it in an analogous manner. Indeed, for such problems we could have a choice initially of making either $U^{(1)}$ or $H^{(1)}$ a pseudo-tensor, and correspondingly for higher-order terms. Such representations are useful in several connections, among them being the representation of the magnetic field in magnetohydrodynamic turbulence.

It is possible to generalize the representation (19) by including an independent random processes at each instant of time as well.[28] Saffman[35] has used a representation of this type to treat particle diffusion problems.

An interesting application of these representations can be found in the study of the approach of processes to Gaussianity following the central limit theorem. Con-Consider, for example, a non-Gaussian process described by the quadratic white noise process,

$$Y(t) = \int_0^t y^{(2)}(t')\, dt'$$

with

$$y^{(2)}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(t - t_1, t - t_2)\, H^{(2)}(t_1, t_2)\, dt_1\, dt_2 \tag{24}$$

and with $H^{(2)}$ defined in (14). The problem considered here is such that the time series $y^{(2)}$ is statistically stationary. We expect by the central limit theorem that $Y(t)$ will become Gaussian as t becomes large, and indeed this will be seen to be the case. We suppose that $Q(t_1, t_2)$ approaches zero for $|t_1|, |t_2| \gg t_0$, with t_0 the correlation time for the process $y^{(2)}$. To check for Gaussianity, consider the flatness factor

$$\text{F.F.} = \langle Y^4 \rangle / \langle Y^2 \rangle^2 \tag{25}$$

Using (24), we find for the variance

$$\begin{aligned} \langle Y^2 \rangle = \int_0^t \int_0^t dt'\, dt'' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(t' - t_1, t' - t_2)\, Q(t'' - t_3, t'' - t_4) \\ \times \langle H^{(2)}(t_1, t_2)\, H^{(2)}(t_3, t_4) \rangle\, dt_1\, dt_2\, dt_3\, dt_4 \end{aligned} \tag{26}$$

Averages of functionals of the type needed in (26) can be found by taking all combinations of δ functions but excluding those δ functions which involve arguments from a single functional (see Reference 19). For example, here

$$\langle H^{(2)}(t_1, t_2) H^{(2)}(t_3, t_4) \rangle = \delta(t_1 - t_3)\, \delta(t_2 - t_4) + \delta(t_1 - t_4)\, \delta(t_2 - t_3) \tag{27}$$

Using the symmetry in the arguments [similar to (21)] of the nonrandom kernel Q, we find

$$\langle Y^2(t)\rangle = \int_0^t dt' \int_{t'}^{t-t'} R(\tau)\, d\tau \tag{28}$$

with

$$R(\tau) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\tau_1 + \tau, \tau_2 + \tau)\, Q(\tau_1, \tau_2)\, d\tau_1\, d\tau_2$$

where we have changed the inner variables of integration. Here $R(\tau)$ is easily found from (24) and (27) to be the correlation function for $y^{(2)}$. We find that (28) can be written

$$\langle Y^2(t)\rangle = t\int_{-\infty}^{\infty} R(\tau)\, d\tau + O(R(0)\, t_0^2) \tag{29}$$

where the error is obtained from the ends of the interval in the first integral of (28), i.e., for t' near zero or t. The relative error is seen to be of order t_0/t. (This result can be made quantitative of course for particular, individual forms of the correlation function.) The result is the familar one obtained for the sum of nearly independent processes, in this case non-Gaussian. We can continue with the calculation of the fourth-order moment. The algebra won't be presented here. One obtains for (25)

$$\text{F.F.} = 3 + O(t_0/t) \tag{30}$$

This is the expected result for a nearly Gaussian process. It is evident that the result depends critically upon having a process $y^{(2)}$ which becomes uncorrelated for sufficiently great time delay. Higher-order moments can be found to be related to the variance as in a Gaussian distribution. This establishes the Gaussianity of the process Y at a given time. One might wonder whether the joint distribution also approaches the Gaussian form. For example, we see from (24) that $\dot{Y}$ is equal to the original non-Gaussian process, and thus could not be expected to have Gaussian characteristics. However, it is easily shown that the variance of $\dot{Y}$ is not proportional to t as is the variance of Y [see (29)]. Hence, non-Gaussian joint characteristics form higher-order parts of the process.[2]

We consider now the moments of log-normal processes. In general of course the calculation of moments of functions of Gaussian processes is difficult. There is one special function, the exponential, for which (as is well known) this is simple. Consider a random process, here taken for simplicity to be statistically stationary,

$$u(t) = \int_{-\infty}^{\infty} P(t - t')\, H^{(1)}(t')\, dt' \tag{31}$$

and consider in turn the random process y defined as follows:

$$y = e^{u(t)} \tag{32}$$

[2] The author is indebted to Professor G. K. Batchelor for this observation.

Then it is easily shown that the average of y is given by

$$\langle y \rangle = e^{\frac{1}{2}\langle u^2 \rangle} \tag{33}$$

Averages of polynomials in y are easily found from these expressions.

To complete the discussion of this section we comment on a characteristic of representations of the type presented here, i.e., they are realizable. To see this, consider the stationary process $y(t)$ given by the first two terms of (17). We know, for example, from stochastic theory that the energy spectrum of the process y must be positive definite. We can construct the energy spectrum by first of all constructing the correlation function for y using the properties of the functionals already given and finding the energy spectrum from this function by taking the Fourier transform. The result is

$$E(\omega) = (2\pi)^{-1}\left[|\, P(\omega)|^2 + (1/\pi)\int_{-\infty}^{\infty} |\, Q(\omega_1\,,\, \omega - \omega_1)|^2\, d\omega_1\right] \tag{34}$$

where P and Q are Fourier transforms of the kernels in (17). Clearly, the energy spectrum is positive. Indeed, all such spectra for this purpose would be positive, as a result of the fact that the representation (17) is realizable: it represents a realizable random process.

3. APPLICATION TO ELECTRICAL SYSTEMS AND FILTERS

References to work of the kind discussed in this section have been given above. In addition to those references, there is a large number of technical reports relating to the use of these functionals in the treatment of filter problems which have been issued from (among other institutions) the Research Laboratory of Electronics, Massachusetts Institute of Technology.

We begin by considering a linear process, in particular, one driven by a statistically stationary forcing term, which process has run long enough so that it has become stationary. Consider the following problem:

$$R\left(\frac{d}{dt}\right) y(t) = f(t) \tag{35}$$

where $f(t)$ is a statistically stationary random process and R is a polynomial function with constant coefficients. We suppose that the process has been running long enough so that $y(t)$ is also statistically stationary, and of course we assume that the polynomial is such that the process is stable. We use a representation like (17) with P, Q,... used to represent $y(t)$, and p, q,... to represent the given kernels used to represent the process $f(t)$. Then, using the statistical orthogonality of the functionals defined in (14), we obtain the following connections between the given quantities and the quantities P, Q,...:

$$\begin{aligned} R\left(\frac{d}{dt}\right) P(t) &= p(t) \\ R\left(\frac{d}{dt}\right) Q(t - t_1\,,\, t - t_2) &= q(t - t_1\,,\, t - t_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{36}$$

Take Fourier transforms to obtain the result

$$\begin{aligned} P(\omega) &= p(\omega)/R(i\omega) \\ Q(\omega_1, \omega_2) &= q(\omega_1, \omega_2)/R(i(\omega_1 + \omega_2)) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{37}$$

and then substitute in (34) to obtain the energy spectrum for the process $y(t)$

$$E_y(\omega) = E_f(\omega)/|\, R(i\omega)|^2 \tag{38}$$

with E_f of the spectrum of f.

Consider now a nonlinear system driven by the idealized white noise process, $H^{(1)}(t)$. Suppose that the system is stable and has operated for a sufficient time so that the response $y(t)$ is statistically stationary. Lee and Schetzen[26] have discussed a cross-correlation technique for the determination of the kernels in the expansion of the process $y(t)$, that is, the determination of the kernels P, Q,... in (17). In our notation we have $H^{(1)}$ as an input to a nonlinear circuit with an output $y(t)$. Accepting the ergodic hypothesis, we have for the first kernel

$$\begin{aligned} T^{-1}\int_0^T H^{(1)}(t-\tau)\, y(t)\, dt &= \langle H^{(1)}(t-\tau)\, y(t)\rangle \\ &= P(\tau) \end{aligned} \tag{39}$$

where T is a sufficiently long interval of time. It is noted that we take the cross-correlation of the output with the input signal alone in order to obtain the coefficient of the Gaussian term in the random process expansion. Proceeding now to the second kernel, we have

$$T^{-1}\int_0^T H^{(2)}(t-\tau_1, t-\tau_2)\, y(t)\, dt = \langle H^{(2)}(t-\tau_1, t-\tau_2)\, y(t)\rangle \tag{40}$$

or, from (17),

$$Q(\tau_1, \tau_2) = (2T)^{-1}\int_0^T H^{(1)}(t-\tau_1)\, H^{(1)}(t-\tau_2)\, y(t)\, dt \tag{41}$$

In (41) we have dropped a bounded fluctuating term, remembering that the Dirac delta function here is replaced where necessary by a quantity proportional to T. From (41) we note that the second-order kernel is obtained by multiple cross-correlation of the output signal with an expression quadratic in the input signal. This process can be continued in an analogous way to higher-order terms. Following Lee and Schetzen, we have here a method for computing directly the Gaussian and various higher-order non-Gaussian terms in the output of a nonlinear circuit (or system) when the circuit is driven by a white noise process. We must of course have access to the input process itself in order to carry out these correlations.

4. APPLICATION TO FLUID MECHANICS

There is a growing body of work devoted to the use of these stochastic representations for the treatment of fluid flow problems, in particular turbulence problems. One of the outstanding characteristics of turbulent phenomena is the fact that velocity fluctuations at a single point have probability distributions which are nearly Gaussian.[2] If a process is to be Gaussian, all odd moments must vanish of course. In Fig. 1 we show the results of experiments performed by Stewart[37] involving fully developed turbulence in a laboratory wind tunnel. The function h is a normalized triple-moment as measured in the turbulent flow under the conditions shown, M is the mesh size of a grid which was inserted in the flow, and x is the distance downstream from the grid position. It is seen that the triple-moment in these measurements is of the order of a few percent. Uberoi[39] has measured even moments in turbulent flow and has compared relationships among them with relationships to be expected for a Gaussian process. He found nearly Gaussian relationships to within 10–15%, the size of the errors in the experiment. On the basis of much experimental information of this type a number of people have formulated theories based upon nearly-Gaussian characteristics. Earlier work involved relationships among the moments, in particular, assumptions of a zero fourth cumulant (Proudman and Reid,[34] Chandrasekhar[10]. Ogura[32] calculated results for the formulation of Proudman and Reid. He found that the solutions for the particular case which he examined were unacceptable. It should be emphasized that Ogura used an exponential form of initial energy spectrum which is known to be far from actual fluid equilibrium spectra. It seems on the basis of later work that nearly-Gaussian theories will not converge properly for initial values too far from equilibrium form.[21]

There is a large body of more recent work involving the use of Wiener functionals (sometimes called Wiener–Hermite functionals). This work has progressed in two

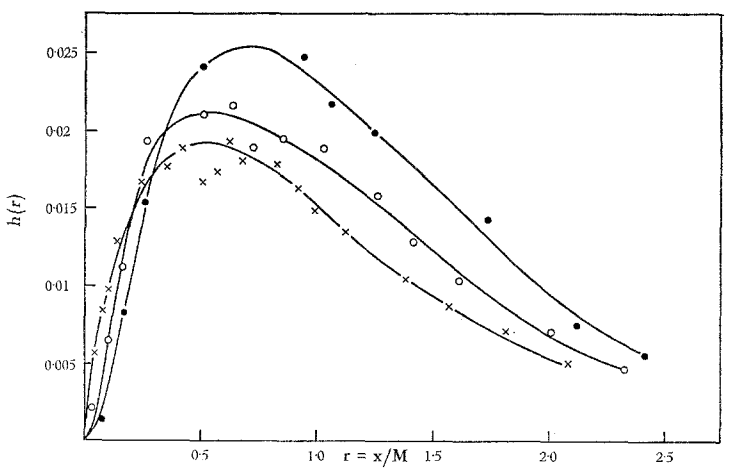

Fig. 1. $x/M = 30$. Black circles: $U = 620$ cm/sec; $M = 1.27$ cm; $R_M = 5300$. Light circles: $U = 620$ cm/sec; $M = 5.08$ cm; $R_M = 21{,}200$. Crosses: $U = 1240$ cm/sec; $M = 5.08$ cm; $R_M = 42{,}400$ (after Stewart[37]).

main groupings. On the one hand much effort has been devoted to model equations similar to the actual fluid Navier–Stokes equation. There has been particular emphasis upon the Burgers model.[6] This model equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - v\frac{\partial^2 u}{\partial x^2} = 0 \tag{42}$$

In statistical work one ordinarily adopts an ensemble of random initial values and traces the statistics of the process through later time. This model equation possesses a known closed-form solution for the initial-value problem (Hopf[18], Cole[12]). The model equation (42) is in certain ways similar to the physical Navier–Stokes equation, but of course the model equation is much simpler although still possessing the characteristic quadratic nonlinearity of many fluid mechanics problems. All of these characteristics taken together have led to considerable interest in the model equation (42). Of course for statistical problems it is necessary to have some numerical results—they might be called the results of numerical experiments. These experiments have been performed by Jeng[21] and Hirschsohn.[17] Numerical experiments have also been reported in some of the other references given here. Moomaw[30] has examined the Gaussianity of this model equation using the closed-form solution. Warming[42] and Benton [3] have considered the statistical characteristics of special solutions of (42). Thomas[38] and Jeng[22] have considered analogous equations for models of magnetohydrodynamic turbulence.

Meecham and Siegel[27] have employed the Wiener representation described above to treat the random-initial-value problem for the Burgers model equation (42). The first two terms were used in the treatment, the function u being written in the form

$$\begin{aligned} u(x) &= u^{(1)}(x) + u^{(2)}(x) \\ &= \int K^{(1)}(x - x_1)\, H^{(1)}(x_1)\, dx_1 + \iint K^{(2)}(x - x_1, x - x_2)\, H^{(2)}(x_1, x_2)\, dx_1\, dx_2 \end{aligned} \tag{43}$$

Equation (43) is then substituted in (42), multiplied successively by $H^{(1)}$ and $H^{(2)}$, and averaged to obtain relationships between the unknown kernel functions $K^{(1)}$ and $K^{(2)}$. In the work under discussion, an approximation was used of the form

$$u^{(2)}(x) = -u^{(1)}\frac{\partial u^{(1)}}{\partial x} \tag{44}$$

(In Reference 27 the approximation was stated in different form but was equivalent to this). The energy equation was then constructed using these forms and integrated for various initial values of the spatial energy spectrum function. Results are shown in Fig. 2. The correlation function and energy spectrum function are defined by

$$Q(r, t) = \langle u(x, t)\, v(x + r, t) \rangle$$

and

$$E(k, t) = \int e^{ikr}\, Q(r, t)\, dr \tag{45}$$

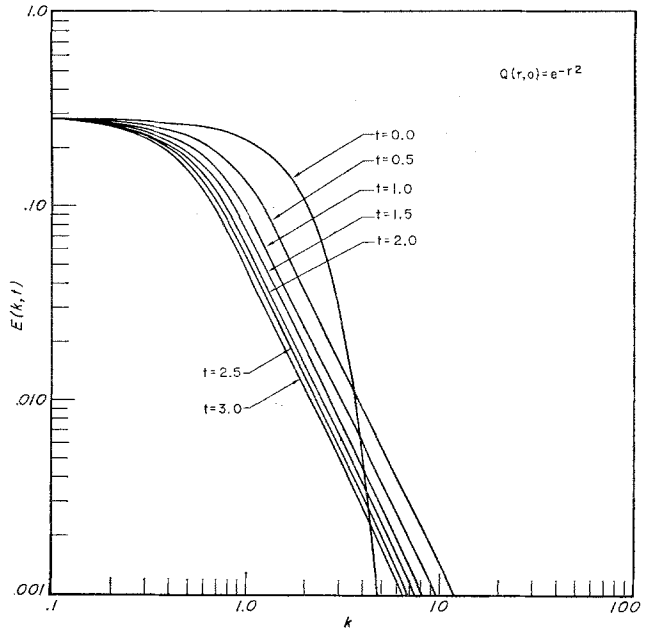

Fig. 2. Log–log plot of the energy spectrum function for the Burgers model equation using the first two terms of the Wiener representation.

The energy spectrum function approaches the form k^{-2}, which is known to be the correct result from numerical experiments already cited. Later work without the assumption (44) (Orszag and Bissonnette,[33] Meecham and Su[29]) has pointed up certain difficulties which occur when the approximation (44) is removed. For Reynolds numbers (uL/ν, where L is the initial scale for the process) of 5 the total energy as given by the first two Wiener functionals follows the numerical experiments quite well for moderate times. This is an interesting result for we know that for such Reynolds numbers the process is strongly nonlinear. However, for larger Reynolds numbers the results of a two-term theory become progressively less satisfactory. It is probable that it will be necessary to use time-dependent white noise processes to adequately represent the nonlinear Burgers model when the process is intensely nonlinear.

Some work has been done on the use of the random representation discussed here in connection with real fluid turbulence. This work has been restricted to statistically homogeneous and isotropic turbulence which is decaying. Again, the first two terms of the Wiener representation were used.[28] Equations are obtained, as for the treatment of the Burgers model just described, connecting the kernels of the first two terms of the Wiener representation. They are somewhat complicated and will not be given here. Initial values for the kernels were obtained from laboratory experiments.[37] The kernel equations were then integrated numerically without further approximation. The normalized triple velocity correlation and energy spectrum function were calculated from these kernels following methods already set forth,

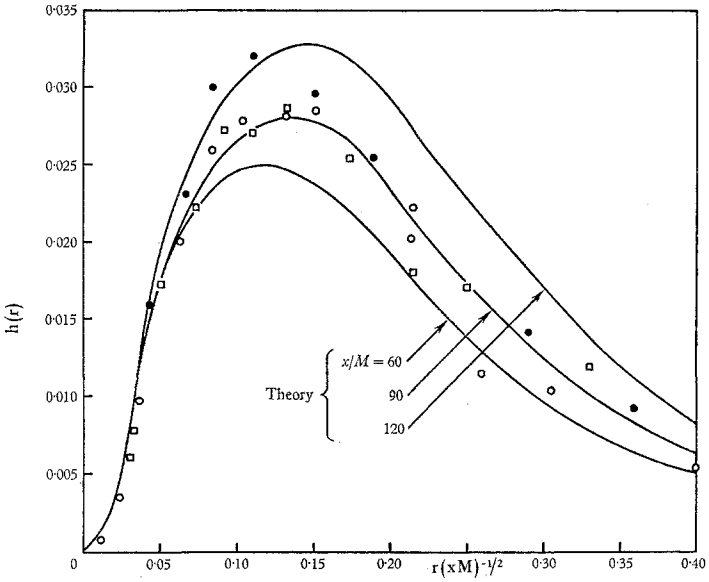

Fig. 3. Triple velocity correlation. Light circles: $x/M = 60$. Squares: $x/M = 90$. Black circles: $x/M = 120$ (after Stewart[37]).

and the results of these calculations were then compared with the results of laboratory experiments for the decaying, homogeneous turbulence. Results for these later times are shown in Fig. 3 for the normalized triple correlation. In this connection x/M may be thought of as the time which has elapsed from the initial turbulence generation process. In Fig. 3 the curves represent the theoretical calculation based on the first two terms in the Wiener expansion. The other symbols represent indicated values for the experiments. For simplicity, we have here modified the abscissae. Stewart actually used a slightly different form in his plots. The results for the energy spectrum function are similar. It can be seen from Fig. 3 that the representation of the experiments is less than perfect. However, the results are encouraging, and it is likely that they could be improved if even more care were taken with the determination of initial values.

Some recent effort has been devoted to various attempts involving the time transformation of the white noise process. It is likely that these attempts will lead to improved forms for the representation of time-decaying processes. For various expositions of this problem the reader is referred to the work of Bodner,[5] Clever,[11] and Canavan and Leith.[9]

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